An Analysis of the Maximality Paradox

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Definition of symbols

Propositional logic

$p,\ q,\ r$	Elementary propositions
1	True
0	False

Natural deduction

$P,\ Q,\ R$	Propositions
∀	Universal quantifier
∃	Existential quantifier

Logical symbols

∧	Conjunction
∨	Disjunction
→	Conditional
↔	Biconditional
¬	Negation
⊢	Proves

Mathematical symbols

x^y	Power, x to the power of y

Standard set theory symbols

$P(A)$	The powerset of A
$A = \{a, b, c, \cdots x, y, z\}$	$a, b, c, \cdots x, y, z$ are the elements of set A
$\{B \mid B \subseteq A\}$	B, such that B is a subset of A
$\emptyset$	The null set
$A = \{a, b, \{a, b\}\}$	A has three elements: a, b, the set whose elements are a and b

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An Analysis of the Maximality Paradox

Symbolic logic is first-order logic. As such, it cannot translate nor address paradoxes. While paradoxes cannot be translated, the arguments themselves that lead to paradoxical conclusions can be translated and analysed.

Argument

I will be analysing the "The Maximality Paradox" as outlined by Ciprotti (2011).

Ciprotti notes that there is no agreed-upon standardisation of The Maximality Paradox, as it has no single attribution. Robert Adams is credited as the first to put it in print, while Martin Davies is credited as the first to formulate it in detail.

Furthermore, the argument in question has no agreed-upon standardised name. It is called The Maximality Paradox by Ciprotti because they think it is appropriate.

The argument Ciprotti outlines is Robert's original formulation, made rigorous by Divers.

The argument is grounded in set theory, which is then expanded and generalised out of set theory (Ciprotti, 2011). However, the extensions are not covered in their (Ciprotti's) paper.

Ciprotti also notes there is 1 hidden assumption: "abstract entities [...] exist". In short, second-order entities and higher-order entities exist.

This paradox examines the nature of "possible worlds" as "world-stories" — "sets of propositions that are both (i) consistent and (ii) maximal collections thereof." Ciprotti (2011) defines the terms (i) "consistent" and (ii) "maximal" as such: (i) "a set A is consistent if and only if it is possible for its members to be jointly true" and (ii) "a set A is maximal if and only if, for every proposition p, either A includes p or A includes the contradiction of p". They (Ciprotti) then note that "such two conditions seem constitutive of the notion of a possible world: a possible world ought to be possible, that is, a contradiction-free entity" and "a possible world ought to be maximal".

In standard argument form, The Maximality Paradox is as such (Ciprotti, 2011):

1. Abstract entities exist.
2. The power set of A, symbolised as P(A), is the set of all subsets of a set A. Thus, P(A) is short for $\{B \mid B \subseteq A\}$. P(A) has 2^n members if A has n members. (Example: suppose that A = {1, 2, 3}. Hence, P(A) = {A, {1, 2}, {1, 3}, {2, 3}, {1}, {2}, {3}, Ø}.)
3. Theorem (so-called “Cantor’s Theorem,” CT): For any set A, every subset of A is smaller than P(A). (Emphasis on “every” because every set A is a subset of itself.)
4. The Proposition Assumption, PA: For each set Ai that is a member of P(A), a proposition p exists that is about that set, namely, the proposition that Ai is a set; if Ai ≠ Aj , then the proposition that Ai is a set and the proposition that Aj is a set are different propositions.
5. Standard set theory is true

P1. There is a maximally consistent set S of propositions (assumption for reductio).

P2. For each set Si that is a member of P(S), there is the proposition p that Si is a set (Proposition Assumption).

P3. For each such p, either p is an element of S or p is not an element of S (definition of maximality condition).

P4. S includes at least as many propositions as there are elements in P(S) (P2, P3).

P5. S is a subset of S (standard set theory).

P6. S has a subset that is at least as large as P(S) (P4, P5).

P7. S has no subset as large as P(S) [CT].

C1. There is no maximal consistent set S of propositions (reductio, P1– P7).

Definitions of one-place and two-place predicates

The following are the definitions of the one-place and two-place predicates that are used in my translation.

A: $abstract$

C: $Consistent$

E: $exists$

Exy: $x\ is\ an\ element\ of\ y$

Ixy: $x\ contains\ y$

Lxy: $x\ is\ at\ least\ as\ large\ as\ y$

Nxy: $x\ has\ at\ least\ as\ many\ propositions\ as\ y\ has\ elements$

M: $Maximal$

P: $the\ proposition\ that\ states\ \exists y_1(Sy_1)\ is\ a\ set$

Pxy: $x\ is\ the\ power\ set\ of\ y$

S: set

Uxy: $x\ is\ a\ subset\ of\ y$

$Q_{small}xy$: $x\ is\ cardinally\ smaller\ than\ y$

$Q_{equal}xy$: $x\ is\ cardinally\ equal\ than\ y$

$Q_{big}xy$: $x\ is\ cardinally\ bigger\ than\ y$

Natural deduction translation

My translation of Ciprotti's Maximality Paradox is as such:

1. $\exists x(Ax \wedge Ex)$
2. $\exists y_1 \exists y_2 (Py_2 y_1)$
3. $\exists y_1 \forall y_3 \forall y_2 (Sy_1 \wedge Q_{small} y_3 y_2)$
4. $\forall y_1 \exists x(Sy_1 \rightarrow Px)$
5. Standard set theory is true

P1. $\exists y_1 (My_1 \wedge Cy_1)$

P2. $\forall y_1 \exists y_2 \exists x(Iy_2 y_1 \rightarrow Px)$

P3. $\exists x \forall y_1 (Px \wedge Sy_1 \wedge (Epy_1 \vee \neg Epy_1))$

P4. $\forall y_1 \forall y_2 (Ny_1 y_2)$

P5. $\forall y_1 (Uy_1 y_1)$

P6. (3.) $\wedge$ (4.) $\rightarrow \forall y_1 \exists y_3 (Sy_1 \rightarrow (Ly_3 y_1 \wedge Uy_3 y_1))$

P7. (3.) $\rightarrow (\forall y_1 \forall y_2 \forall y_3 ((Py_2 y_1 \wedge Uy_3 y_1) \rightarrow (\neg Q_{equal} y_3 y_2 \wedge \neg Q_{big} y_3 y_2)))$

C1. $\neg \forall y_1 (My_1 \wedge Cy_1)$

Propositional logic translation

1. p
2. Definition of power set

3. Cantor's theorem

4. Propositional assumption

5. Standard set theory is true

P1. q

P2. r

P3. $s \vee \neg s$

P4. $r \wedge s \rightarrow t$

P5. u

P6. $t \wedge u \rightarrow v$

P7. $\neg t$

C1. $\neg q$

Natural deduction analysis

The problem with translating The Maximality Paradox into natural deduction is that no translation will perfectly capture the mathematical nuances the statements hold. For example, x refers to entities, which is technically different from sets. $y_1,\ y_2\ and\ y_3$ share the same variable class, but are fundamentally different mathematical objects (sets, power sets and subsets). The mathematical relation each object has with each other is better expressed mathematically rather than in a two-place predicate. A two-place predicate can capture the cardinality of something, and whether something is an element of another, but in doing so will isolate the relation into separate predicates. These predicates are unable to interact with each other, and thus will not produce desirable results.

Furthermore, (P4.) is not a measurement of the cardinality of a set. Rather, it is mapping the elements of one set to another. This builds into (P6.) as $(P4.) \rightarrow (P6.)$. However, this contradicts (3.), which introduces the necessary R.A.A. that disproves the principle assumption, (P1.). Thus, the two-place predicate captures part of (P4.)'s meaning, but not all of its qualities.

It is immediately obvious that it is impossible to go from (P1.) to (C1.). (P2.) through (P7.) does not contain the element "$My_1 \wedge Cy_1$". Thus, there is no relation with (P1.) ad (C1.) with (P2.) through (P7.) within natural deduction. In other words, natural deduction states no claim on the validity of this argument; It is not possible to guarantee that the conclusion follows the premise.

Propositional logic analysis

The translation using propositional logic is more promising as it can capture the logical relation (P1.) has with every other premise, as well as the conclusion, (C1.).

However, the argument in propositional logic is incomplete. It has hidden premises that must be included to be rigorous. 2 premises should be added, (1.) $q \vee \neg q$ and (2.) the logical relation q has with every other proposition (r, s, t, u, v). Additionally, (P6.) is two-folded. It has both a statement and the justification for that statement. It really says two things: v and $t \wedge u \rightarrow v$. Likewise, (P4.) really says t and $r \wedge s \rightarrow t$.

I deduce the logical relation q has with every other proposition to be $(p \wedge (t \wedge u \rightarrow v)) \rightarrow q$. The heart of the argument is only $(t \wedge u \rightarrow v) \leftrightarrow q$ as p is taken to be true.

Therefore, the argument should be:

P1.	$q \vee \neg q$	Things are either maximal and consistent or they are not
P2.	q	Assumption for R.A.A.
P3.	$q \rightarrow (r \wedge s \wedge u)$	Logical relation of q
P4.	$r \vee \neg r$	Proposition Assumption
P5.	$s \vee \neg s$	Definition of maximality condition
P6.	$(r \wedge s) \rightarrow t$	n(Propositions) of S = n(Elements) of P(S)

P7.	u	Standard set theory
P8.	$(((r \wedge s) \rightarrow t) \wedge u) \rightarrow v$	S has a subset that is at least as large as P(S)
P9.	$\neg v$	Cantor's Theorem
C1.	$\neg q$	Conclusion; A thing is not maximal and consistent

To arrive at the conclusion, the following manipulation can be done.

$\vdash \neg q$		
Assumption	Formula	Rule
	1. $q \vee \neg q$	
	2. q	
	3. $q \rightarrow (r \wedge s \wedge u)$	
	4. $r \vee \neg r$	
	5. $s \vee \neg s$	
	6. $(r \wedge s) \rightarrow t$	
	7. u	
	8. $(((r \wedge s) \rightarrow t) \wedge u) \rightarrow v$	
	9. $\neg v$	
[2, 3]	10. $r \wedge s \wedge u$	2, 3 M.P.
[2, 3]	11. $r \wedge s$	10 Simp.
[2, 3, 6]	12. t	6, 10 M.P.
[2, 3, 6]	13. $((r \wedge s) \rightarrow t)$	11, 12 C.P.
[2, 3, 6, 7]	14. $((r \wedge s) \rightarrow t) \wedge u)$	7, 13 Conj.
[2, 3, 6, 7, 8]	15. v	8, 14 M.P.
[2, 3, 6, 7, 8, 9]	16. $v \wedge \neg v$	9, 15 Conj.
[3, 6, 7, 8, 9]	17. $\neg q$	16 R.A.A.

Thus, this argument is valid within this formulation. It should be noted that the logical relation of q is debatable.

Ciprotti argues that for a set to be maximal and consistent, it needs to fulfil a strict set of mathematical requirements. If just one of these requirements is not met, the set in question is not maximal and consistent. This idea can be translated as $q \leftrightarrow (r \wedge s \wedge u)$.

The first way ($q \rightarrow (r \wedge s \wedge u)$) is used within the proof of $\neg q$. The second way ($(r \wedge s \wedge u) \rightarrow q$) is not used as it will lead to a dead end. Nothing can be said because the premise is false. That does not mean the conclusion is necessarily false.

Does this mean the argument presented by Ciprotti is invalid because there exists a translation which invalidates the argument? Or is it valid because there exists one translation which is valid? It is troubling because we can ignore translations that are malicious or inaccurate, but we cannot ignore equally sound translations that disagree on an argument's validity. In Ciprotti's argument, I suppose the natural deduction translation can be written off as an unsound translation. However, there is no definitive proof against a sound formulation within natural deduction that leads to an invalid proof.

Framing the The Maximality Paradox in symbolic logic form, it is paradoxical because both q and $\neg q$ are true. We can create sets that are maximal and consistent, which then logically concludes that sets that are not maximal and consistent exist.

While The Maximality Paradox is essentially a logical paradox, the argument that q derives $\neg q$ is contained within first-order logic.

It should also be noted that in my analysis, I assume that mathematical objects can be empirically measured. This makes it possible to translate mathematical statements into well-formed formulas (w.f.f.s) within symbolic logic. Given that this touches on the philosophy of mathematics, it is beyond the scope of this paper.

The problem with translating mathematical statements within symbolic logic is the lack of mathematical manipulation. Finding contradictions within axiomatic deduction is far easier. Statements such as (P4.) "x has at least as many propositions as y has elements" could be translated within axiomatic deduction as (P4.) $\exists x \exists y((y \mid P(x)) \land (|x| = |y|))$. Thus, if one finds out that $|x| \neq |y|$ is empirically true, one immediately disproves the statement "x has at least as many propositions as y has elements".

It should also be noted that logic can describe itself. For example, we can translate the statement "Symbolic logic is complete or symbolic logic is incomplete" as $p \lor \neg p$ and then empirically determine the truth value of p, reaching the ultimate conclusion that symbolic logic itself is either complete or incomplete. However, there are several concerns. Paradoxical conclusions can be reached, of which symbolic logic is unable to translate or analyse.

Conclusion

The Maximality Paradox is only conditionally valid within symbolic logic. It should be empirically true within mathematics as it is sound mathematically. However, I speculate a translation of the argument exists such that it concludes that the paradox is invalid. Thus, the formal proof of validity must be made rigorous to arrive at a definite conclusion.

Furthermore, The Maximality Paradox should be standardised as a (meta-)mathematical paradox. Currently, there is no one true formulation of the paradox, nor is there even consensus on the title. The ambiguity of validity may be caused by the inherent ambiguity of the paradox itself.

Several insights can be drawn after attempting to conclude whether The Maximality Paradox is valid or not. First, things that are called paradoxes can still be translated in symbolic logic if the key argument itself is not paradoxical. Second, translating mathematical statements into symbolic logic is challenging as there is much room for interpretation. Third, should mathematical statements be translated into symbolic logic? Natural deduction conflicts with axiomatic deduction, which is better equipped to deal with mathematical statements. Elementary propositions are better suited as it relies on empirically-measurable statements, which is what The Maximality Paradox touches on.

Further discussion must be done before coming to a definitive conclusion regarding The Maximality Paradox's validity.

References

Ciprotti, N. (2011). The Maximality Paradox. In M. Bruce & S. Barbone (eds.), *Just the Arguments: 100 of the Most Important Arguments in Western Philosophy* (pp. 141-7). Wiley-Blackwell. https://doi.org/10.1002/9781444344431.