Take home assignments for Set Theory and Logic

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Assignment 1 (first-order logic)

Q1(a). Show that $\Gamma \cup \{\alpha\} \vDash \varphi$ iff $\Gamma \vDash (\alpha \rightarrow \varphi)$.

note. The question is asking us to show that $(\Gamma \cup \{\alpha\} \vDash \varphi) \leftrightarrow (\Gamma \vDash (\alpha \rightarrow \varphi))$. In other words, if we set proposition p: $(\Gamma \cup \{\alpha\} \vDash \varphi)$, and q: $(\Gamma \vDash (\alpha \rightarrow \varphi))$, then we need to show $p \leftrightarrow q$.

proof.

case 1 (⇒). Assume p is true. That is, assume some premise α is not in Γ, and that φ is a tautology in $\Gamma \cup \{\alpha\}$. That is, φ is always true in our proof system $\Gamma \cup \{\alpha\}$. Since we have $\Gamma \cup \{\alpha\} \vDash \varphi$, then by the Completeness theorem, $\Gamma \cup \{\alpha\} \vdash \varphi$ (i.e. every tautology is provable in a propositional logic proof system).

Note that we have	$\Gamma \cup \{\alpha\} \vdash \varphi$
which is the same as	$\Gamma, \alpha \vdash \varphi$
thus, by the deduction theorem	$\Gamma \vdash (\alpha \rightarrow \varphi)$
by the soundness theorem	$\Gamma \vDash (\alpha \rightarrow \varphi)$

therefore, $(\Gamma \cup \{\alpha\} \vDash \varphi) \rightarrow (\Gamma \vDash (\alpha \rightarrow \varphi))$.

note. We did not need to assume that α is not in Γ, as α could already be part of Γ. Nevertheless, the case above works regardless.

case 2 (⇐). Assume q is true. That is, $\Gamma \vDash (\alpha \rightarrow \varphi)$ is true. That means, $v(\alpha \rightarrow \varphi) = t$. Now, we have two "objects" or "wffs": Γ, and $(\alpha \rightarrow \varphi)$.

$\Gamma \vDash (\alpha \rightarrow \varphi)$	assumption (premise q)
$\Gamma \vdash (\alpha \rightarrow \varphi)$	by Completeness theorem

case 2.1. Suppose α is not in Γ. Then we can simply $\Gamma \cup \{\alpha\}$ by assuming some premise α.

$\Gamma, (\alpha \to \varphi), a \vdash \varphi$ is true, because:

1. $(\alpha \to \varphi)$ premise

2. α premise

3. φ 1, 2 M.P.

therefore, we can conclude $\Gamma \cup \{\alpha\} \vdash \varphi$ if $\Gamma \vDash (\alpha \to \varphi)$.

4. $\Gamma \cup \{\alpha\} \vdash \varphi$ 1, 3 M.P.

5. $\Gamma \cup \{\alpha\} \vDash \varphi$ 4 Soundness theorem

therefore, $\Gamma \cup \{\alpha\} \vDash \varphi$ if $\Gamma \vDash (\alpha \to \varphi)$.

case 2.2. Suppose α is in Γ. Then, we can still $\Gamma \cup \{\alpha\}$ which is Γ. Following the exact same argument as case 2.1, we can conclude $\Gamma \cup \{\alpha\} \vDash \varphi$ if $\Gamma \vDash (\alpha \to \varphi)$.

rejoinder. With case 1 and case 2, we have:

6. $(\Gamma \cup \{\alpha\} \vDash \varphi) \to (\Gamma \vDash (\alpha \to \varphi))$ case 1

7. $(\Gamma \vDash (\alpha \to \varphi)) \to (\Gamma \cup \{\alpha\} \vDash \varphi)$ case 2

8. $((\Gamma \cup \{\alpha\} \vDash \varphi) \to (\Gamma \vDash (\alpha \to \varphi)) \wedge (\Gamma \vDash (\alpha \to \varphi)) \to (\Gamma \cup \{\alpha\} \vDash \varphi))$ 6, 7 Conjunction

9. $(\Gamma \cup \{\alpha\} \vDash \varphi) \leftrightarrow (\Gamma \vDash (\alpha \to \varphi))$ 8 definition of $\leftrightarrow$

This completes my proof.

Q1(b). φ, ψ are tautologically equivalent iff $\vDash (\varphi \leftrightarrow \psi)$.

note. The question is asking us to show that (φ, ψ are tautologically equivalent) $\leftrightarrow$ ($\vDash (\varphi \leftrightarrow \psi)$) for some well formed formulas φ and ψ. Once again, we can set proposition p: φ, ψ are tautologically equivalent and proposition q: $\vDash (\varphi \leftrightarrow \psi)$. Thus, we need to show $p \leftrightarrow q$.

At first glance, this question is strange, because of two reasons. First, "tautologically equivalent" isn't a term explicitly defined in lecture. But I take it that tautological equivalence to be $\leftrightarrow$. Second, if my understanding is correct, then the question reads "Show that $(\varphi \leftrightarrow \psi) \leftrightarrow (\vDash (\varphi \leftrightarrow \psi))$ is true", which seems almost trivial. But there is a subtle significance to this biconditional.

proof.

case 1 ($\Rightarrow$). Assume p: φ, ψ are tautologically equivalent, i.e. $(\varphi \leftrightarrow \psi)$. By assuming that $(\varphi \leftrightarrow \psi)$, we are asserting that for some proof system Γ, $\Gamma \vDash (\varphi \leftrightarrow \psi)$.

case 1.1. Γ does not contain any additional wffs, i.e. it only contains the axioms I-IX and MP. Then $\Gamma \vDash (\varphi \leftrightarrow \psi)$ is really just $\vDash (\varphi \leftrightarrow \psi)$. Therefore, we can conclude $((\varphi \leftrightarrow \psi) \rightarrow (\vDash (\varphi \leftrightarrow \psi))$.

That is,

1. $(\varphi \leftrightarrow \psi)$	assumption (premise p)
2. $\vDash (\varphi \leftrightarrow \psi)$	because our assumptions must be tautological
3. $((\varphi \leftrightarrow \psi) \rightarrow (\vDash (\varphi \leftrightarrow \psi))$	1, 2 MP

case 1.2. Γ contains some additional wffs, including the axioms I-IX and MP. Then, our assumption asserts that in Γ, $(\varphi \leftrightarrow \psi)$ is true. From that, we can conclude $(\Gamma \vDash (\varphi \leftrightarrow \psi))$. Therefore, for some proof system Γ, where $(\varphi \leftrightarrow \psi)$ is in Γ, $((\varphi \leftrightarrow \psi) \rightarrow (\Gamma \vDash (\varphi \leftrightarrow \psi)))$ is in Γ. We can simply restate this as "$((\varphi \leftrightarrow \psi) \rightarrow (\vDash (\varphi \leftrightarrow \psi))$ is in Γ".

Combining case 1.1 and case 1.2, $((\varphi \leftrightarrow \psi) \rightarrow (\vDash (\varphi \leftrightarrow \psi))$ is true.

That is,

4. $(\varphi \leftrightarrow \psi)$ is in Γ assumption (premise p)

5. $\Gamma \vDash (\varphi \leftrightarrow \psi)$ because our assumptions must be tautological

6. $((\varphi \leftrightarrow \psi) \rightarrow (\vDash (\varphi \leftrightarrow \psi))$ is in Γ 4, 5 MP

case 2 ($\Leftarrow$). Assume q: $\vDash (\varphi \leftrightarrow \psi)$. Since $(\varphi \leftrightarrow \psi)$ is a tautology, we can conclude $(\varphi \leftrightarrow \psi)$.

7. $\vDash (\varphi \leftrightarrow \psi)$ assumption (premise q)

8. $(\varphi \leftrightarrow \psi)$ if something is tautological, it exists as an "object" in Γ

9. $((\vDash (\varphi \leftrightarrow \psi)) \rightarrow (\varphi \leftrightarrow \psi))$ 7, 8 MP

rejoinder. Combining case 1 and 2:

10. $((\varphi \leftrightarrow \psi) \rightarrow (\vDash (\varphi \leftrightarrow \psi)) \wedge ((\vDash (\varphi \leftrightarrow \psi)) \rightarrow (\varphi \leftrightarrow \psi))$ 3, 9 Conjunction

11. $((\varphi \leftrightarrow \psi) \leftrightarrow (\vDash (\varphi \leftrightarrow \psi))$ 10 definition of $\leftrightarrow$

This completes my proof.

Q2. (Duality) Let α be a propositional wff whose only connectives are $\vee$, $\wedge$ and $\neg$. Let α^* be the formula obtained from α by interchanging each $\vee$ with $\wedge$ (and every $\wedge$ with $\vee$), and replacing each propositional symbol with its negation. Show that α^* is tautologically equivalent to $\neg\alpha$. You should use induction.

proof by structural induction.

base case. Let $\alpha = p$. Note that it has no connective symbols. To obtain α^*, we must go through three steps as outlined in the question. Step 1: replace every $\vee$ with $\wedge$. We obtain p. Step 2: replace every $\wedge$ with $\vee$. We obtain p. Step 3: replace every propositional symbol with its negation. We obtain $\neg p$. Thus, $\alpha^* = \neg p$. Note that the negation of α is $\neg\alpha$, which is $\neg p$. Evidently, $\neg p = \neg p$, therefore $\neg\alpha = \alpha^*$.

note. All wff are constructed in a recursive manner. For example, $p \wedge q$ is a parent of two children: p, q. The children are connected via $\wedge$, a binary connective symbol. This example also covers $\vee$ by replacing $\wedge$ with $\vee$. Another example: $\neg p$ is a parent of one child, p, which is "connected" by $\neg$, a unary symbol.

case 2 ($\wedge$). Let $\alpha = p \wedge q$.

We want to obtain α^* and $\neg\alpha$. To do so, we apply the operations relevant to α^* and $\neg\alpha$ and compare the results.

1. $p \wedge q$	this is α
2. $p \wedge q$	replace every $\vee$ with $\wedge$
3. $p \vee q$	replace every $\wedge$ with $\vee$
4. $\neg p \vee \neg q$	replace every propositional symbol with its negation

we obtain $\alpha^* = \neg p \vee \neg q$.

Next, we negate α.

5. $p \wedge q$	this is α
6. $\neg(p \wedge q)$	$\neg\alpha$

7. $\neg p \lor \neg q$ De Morgan's law

We obtain $\neg\alpha = \neg p \lor \neg q$.

Comparing (4.) and (7.), we see that they are exactly the same. Therefore, we conclude that $\neg\alpha = \alpha^*$.

case 3 ($\lor$). Let $\alpha = p \lor q$.

We follow the same operations and logic as before.

8. $p \lor q$ this is α

9. $p \land q$ replace every $\lor$ with $\land$

10. $p \land q$ replace every $\land$ with $\lor$

11. $\neg p \land \neg q$ replace every propositional symbol with its negation

We obtain $\alpha^* = \neg p \land \neg q$.

Next, we negate α.

12. $p \lor q$ this is α

13. $\neg(p \lor q)$ $\neg\alpha$

14. $\neg p \land \neg q$ De Morgan's law

Comparing (11.) and (14.), we see that they are exactly the same. Therefore, we conclude that $\neg\alpha = \alpha^*$.

case 4 ($\neg$). Let $\alpha = \neg p$.

We follow the same operations and logic as before.

15. $\neg p$ this is α

16. $\neg p$ replace every $\lor$ with $\land$

17. $\neg p$ replace every $\land$ with $\lor$

18. $\neg(\neg p)$ replace every propositional symbol with its negation

19. p "Double negation" rule

We obtain $\alpha^* = p$.

Next, we negate α.

20. ¬p	this is α
21. ¬(¬p)	¬α
22. p	"Double negation" rule

Comparing (19.) and (22.), we see that they are exactly the same. Therefore, we conclude that ¬α = α*.

note ("Double negation" rule). ¬(¬p) ≡ p because of its corresponding truth table values. As the first and third columns is exactly the same, ¬(¬p) ≡ p.

(p)	(¬p)	¬(¬p)
1	0	1
0	1	0

Induction step. Through the results of the base case, and cases 2-4, and given that α is constructed recursively, α* will always be tautologically equivalent to ¬α. This is regardless of how complex α is. For example, if α = ¬(p ∨ q), we note that α is either covered by the base case, or cases 2-4. If we let (p ∨ q) = β, then α = ¬(β), which is covered by case 4. β itself is covered by case 3. The propositional symbols themselves, i.e. p, q, are covered by the base case.

This concludes my proof.

Q3. Use H to denote the propositional system consisting of the following three rules:

I. $\vdash A \rightarrow (B \rightarrow A)$

II. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

MP. $A, (A \rightarrow B) \vdash B$

Use this system to prove the theorem $A \rightarrow A$. Hint. The first step is to prove the following theorem: $(A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$.

Note. There are two axioms, and one rule of inference MP. We want to prove the theorem $A \rightarrow A$. We also note that A, B, C are wff, which contains propositional symbols, and could contain connectors, i.e. $\neg$, $\rightarrow$, $\wedge$, $\vee$.

Proof. Let A be A, B be $(A \rightarrow A)$, and C be A. Then,

1. $\vdash (A \rightarrow ((A \rightarrow A) \rightarrow A))$ axiom I, restated
2. $\vdash (A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ axiom II, restated
3. $(A \rightarrow ((A \rightarrow A) \rightarrow A)), (A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)) \vdash ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ 1, 2 MP
4. A assumption
5. $A, (A \rightarrow (B \rightarrow A)) \vdash (B \rightarrow A)$ I, 4 MP

For this result specifically, if we replace B with A, we obtain:

6. $A, (A \rightarrow (A \rightarrow A)) \vdash (A \rightarrow A)$ 5 restated
7. $\vdash A \rightarrow (A \rightarrow A)$ by Deduction theorem
8. $(A \rightarrow (A \rightarrow A)), ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)) \vdash (A \rightarrow A)$ 3, 7 MP

This concludes my proof.

Q4. (First-order logic.) Show that $\{\forall x(\alpha\to\beta), \forall x(\alpha)\} \vDash \forall x(\beta)$.

Want to show. if $\mathscr{A} \vDash \forall x(\alpha\to\beta)$ and if $\mathscr{A} \vDash \forall x(\alpha)$, then $\mathscr{A} \vDash \forall x(\beta)$ for some structure $\mathscr{A}$.

Proof. Suppose we have a structure $\mathscr{A}$.

$\mathscr{A} \vDash \forall x(\alpha \to \beta)[a_1, \ldots, a_n]$ if $x_1, \ldots, x_n$ includes all free variables, and the "p→q" truth table holds in $\mathscr{A}$. Set this as premise 1.

$\mathscr{A} \vDash \forall x(\alpha)[a_1, \ldots, a_n]$ if $x_1, \ldots, x_n$ includes all free variables, and for each $a \in \mathscr{A}$, we have $\mathscr{A} \vDash \forall x(\alpha)[a_1, \ldots, a_n]$. Set this as premise 2.

Given premise 1 and 2, we obtain $\mathscr{A} \vDash \forall x(\beta)[a_1, \ldots, a_n]$ via Modus Ponens. That is, in premise 2, we know that α is true for all x in $\mathscr{A}$. In premise 1, we know that $\alpha \to \beta$ holds for all x in $\mathscr{A}$. Thus, we obtain β is always true for all x in $\mathscr{A}$.

This completes my proof.

Assignment 2 (completeness, compactness, chains)

Q1. Prove these using the formal definition of $\vDash$.

(a) $\{\forall x(\alpha \to \beta), \forall x\alpha\} \vDash \forall x\beta$

proof. Consider some language $\mathscr{L}$ and some structure $\mathscr{A}$, and some Γ, a set of well-formed formulas in first-order logic.

Suppose $\{\forall x(\alpha \to \beta), \forall x\alpha\} \subseteq \Gamma$. (where α, β are some wff in first-order logic.)

Then $\Gamma \vDash \forall x(\alpha \to \beta)$,

and $\Gamma \vDash \forall x\alpha$.

$\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$ Axiom II of Λ

$\{\forall x(\alpha \to \beta), \forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)\} \vdash \forall x\alpha \to \forall x\beta$ by MP rule

$\{\forall x\alpha \to \forall x\beta, \forall x\alpha\} \vdash \forall x\beta$ by MP rule

$\{\forall x(\alpha \to \beta), \forall x\alpha\} \vDash \forall x\beta$ by Soundness theorem □

(b) $\alpha \vDash \forall x\alpha$, if x does not occur free in α.

proof. Consider some α, a wff in first-order logic, which contains a bounded variable x.

Consider some Γ where $\{\alpha\} \subseteq \Gamma$. That is, $\Gamma \vDash \alpha$.

$\alpha \to \forall x\alpha$ Axiom IV of Λ, given that x is not free in α

$\{\alpha, \alpha \to \forall x\alpha\} \vdash \forall x\alpha$ by MP rule

$\Gamma \vdash \forall x\alpha$ since $\{\alpha\} \subseteq \Gamma$

$\Gamma \vDash \forall x\alpha$ by Soundness theorem

Since we have $\Gamma \vDash \alpha$ and $\Gamma \vDash \forall x\alpha$, we conclude that $\alpha \vDash \forall x\alpha$ (by definition of $\vDash$). □

Q2. One of the following is valid, the other is not. Give a proof (from the axioms, and not using the metatheorems) of the valid one:

$$(\forall x\varphi) \to (\exists y\varphi) \quad \text{— (1)}$$

$$\forall x(\varphi \rightarrow (\exists y)\varphi) \qquad \text{— (2)}$$

proof. since $\forall x(\varphi \rightarrow (\exists y)\varphi) \equiv ((\forall x\varphi) \rightarrow (\forall x \exists y\varphi))$ via Axiom III of Λ, we can rewrite equations (1) and (2) to obtain the following:

$$(\forall x\varphi) \rightarrow (\exists y\varphi) \qquad \text{— (3)}$$

$$(\forall x\varphi) \rightarrow (\forall x \exists y\varphi) \quad \text{— (4)}$$

case 1. (4) The RHS of (4) asserts that φ has two bounded variables, x and y. However, on the LHS of (4), x is the only bounded variable in the same φ. Therefore, (4) is invalid.

case 2. (3) It is evident that (3) only has one bounded variable in φ, given that there is only one quantifier on the LHS (universal) and the RHS (existential). That is, if $\forall x\varphi$ is true, that necessitates the existence of a single x where φ is true. Thus, $\exists y\varphi$ is true, as desired. $\square$

Q3. Are the following statements true or false? If the statement is true, prove it (from the axioms and metatheorems, but not using the Completeness Theorem). Otherwise, give a counter-example.

(a) $\Gamma \vdash \alpha \rightarrow (\beta \rightarrow \gamma)$ if and only if $\Gamma \vdash (\alpha \wedge \beta) \rightarrow \gamma$.

claim. This statement is true.

proof.

case 1. ($\Rightarrow$)

Assume $\Gamma \vdash \alpha \rightarrow (\beta \rightarrow \gamma)$. Assume $\alpha \in \Gamma$.

$\{\alpha, \alpha \rightarrow (\beta \rightarrow \gamma)\} \vdash (\beta \rightarrow \gamma)$ MP rule

Assume $\beta \in \Gamma$.

$\{\beta, (\beta \rightarrow \gamma)\} \vdash \gamma$ MP rule

Therefore, $\{\alpha, \beta\} \vdash \gamma$, i.e. $(\alpha \wedge \beta) \rightarrow \gamma$. Thus, $\Gamma \vdash (\alpha \wedge \beta) \rightarrow \gamma$ as desired.

case 2. ($\Leftarrow$)

Assume $\Gamma \vdash (\alpha \wedge \beta) \rightarrow \gamma$. Assume $\alpha, \beta \in \Gamma$.

$(\alpha \wedge \beta)$ Conjunction

$\{(\alpha \wedge \beta), (\alpha \wedge \beta) \rightarrow \gamma\} \vdash \gamma$ MP rule

therefore, $\Gamma \cup \{\alpha\} \cup \{\beta\} \vdash \gamma$

$\Gamma \cup \{\alpha\} \vdash (\beta \rightarrow \gamma)$ Deduction theorem

$\Gamma \vdash \alpha \rightarrow (\beta \rightarrow \gamma)$ Deduction theorem

Combining both cases, $\Gamma \vdash \alpha \rightarrow (\beta \rightarrow \gamma)$ if and only if $\Gamma \vdash (\alpha \wedge \beta) \rightarrow \gamma$ as desired.☐

(b) $\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$ implies that $\Gamma \vdash \alpha \rightarrow \gamma$ and $\Gamma \vdash \beta \rightarrow \gamma$.

claim. This statement is false.

proof. Assume Γ is consistent, i.e. Γ only proves tautologies.

case 1. Assume $\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$.

$\Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$ Deduction theorem

case 2. Assume $\Gamma \vdash \alpha \rightarrow \gamma$, $\Gamma \vdash \beta \rightarrow \gamma$.

$\Gamma \cup \{\alpha\} \vdash \gamma$, $\Gamma \cup \{\beta\} \vdash \gamma$ Deduction theorem

That is, $\Gamma \cup \{\alpha, \beta\} \vdash \gamma$. This implies $\Gamma \cup \{\alpha \wedge \beta\} \vdash \gamma$.

rejoinder. It is clear to see that $\alpha \vee \beta \not\equiv \alpha \wedge \beta$. Therefore, $\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$ does not imply that $\Gamma \vdash \alpha \rightarrow \gamma$ and $\Gamma \vdash \beta \rightarrow \gamma$.

proof by counterexample. Suppose α, γ is true, while β is false. Then, $(\alpha \vee \beta) \rightarrow \gamma$ and $\alpha \rightarrow \gamma$ is true, while $\beta \rightarrow \gamma$ is false. Therefore, $\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$, $\Gamma \vdash \alpha \rightarrow \gamma$, and $\Gamma \nvdash \beta \rightarrow \gamma$. ☐

(c) $\Gamma \vdash \alpha \rightarrow \gamma$ or $\Gamma \vdash \beta \rightarrow \gamma$ implies that $\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$.

claim. This statement is true.

proof. Assume Γ is consistent.

case 1. Assume $\Gamma \vdash \alpha \rightarrow \gamma$.

$\Gamma \cup \{\alpha\} \vdash \gamma$ Deduction theorem

$\Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$ Generalisation

$\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$ Deduction theorem

case 2. Assume $\Gamma \vdash \beta \rightarrow \gamma$.

$\Gamma \cup \{\beta\} \vdash \gamma$ Deduction theorem

$\Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$ Generalisation

$\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$ Deduction theorem

rejoinder. Given case 1 and 2, we have $\Gamma \vdash (\alpha \vee \beta) \rightarrow \gamma$ as desired. □

Q4. Let $\mathscr{L} = \{\leq\}$ be the language with a single binary relation, and let $\mathscr{N} = (\mathbb{N}, \leq^{\mathscr{N}})$ be the natural numbers with the usual ordering. Show that there is a structure $\mathscr{M} = (M, \leq^{\mathscr{M}})$ that satisfies all the same sentences of $\mathscr{N}$ (they are elementarily equivalent, written $\mathscr{M} \equiv \mathscr{N}$), such that $\mathscr{M}$ contains an infinite descending chain of elements: $\ldots \leq^{\mathscr{M}} m_3 \leq^{\mathscr{M}} m_2 \leq^{\mathscr{M}} m_1$.

proof. Let us construct $\mathscr{M}$ such that $M = \mathbb{N} \cup \{\ldots \leq^{\mathscr{M}} m_3 \leq^{\mathscr{M}} m_2 \leq^{\mathscr{M}} m_1\}$, where m_i (for any $i \in \mathbb{N}$) is not in $\mathbb{N}$. In fact, m_i need not be a number. All we know about m_i is that $\ldots \leq^{\mathscr{M}} m_3 \leq^{\mathscr{M}} m_2 \leq^{\mathscr{M}} m_1$ holds. Since "an infinite descending chain of elements" is inexpressible in first-order logic, it is clear to see that $\mathscr{M} \equiv \mathscr{N}$, given the construction of $\mathscr{M}$. Thus, every first-order logic sentence will only be about $\mathbb{N}$, of which both $\mathscr{N}$ and $\mathscr{M}$ contains. Specifically, if Γ is the set of first-order logic sentences about $\mathscr{N}$, assume that Γ is (finitely) satisfiable. Given the construction of $\mathscr{M}$, Γ is also the set of first-order logic sentences about $\mathscr{M}$. As Γ is finitely satisfiable, Γ is satisfiable (by Compactness Theorem). Set $\Gamma_1 = \{\forall x \exists y(x \leq y)\}$. It is evident that $\forall x \exists y(x \leq y)$ is true for all $x, y \in \mathbb{N}$. $\forall x \exists y(x \leq y)$ is still true even if $x, y \in \mathbb{N} \cup \{m_i: i \in \mathbb{N}\}$, or if we treat m_i as a number beyond the naturals (whatever that means). Formally, let $\varphi_{\mathscr{N}}$: all sentences of $\mathscr{N}$ that is satisfiable. Similarly, let $\varphi_{\mathscr{M}}$: all sentences of $\mathscr{M}$ that is satisfiable. Let S: $Th(\varphi_{\mathscr{N}})$. Let $T \subseteq S$ be finite. By the Compactness theorem, T is satisfiable in $\mathscr{N}$. Given how $\mathcal{M}$ is constructed, T is satisfiable in $\mathcal{M}$. Thus, $\varphi_{\mathscr{N}} \equiv \varphi_{\mathcal{M}}$, $\mathrm{Th}(\varphi_{\mathscr{N}}) \equiv \mathrm{Th}(\varphi_{\mathcal{M}})$, and $\mathcal{M} \equiv \mathscr{N}$. $\square$

Assignment 3 (set theory)

Q1. By considering a variation of Russell's paradox, show that $\mathscr{P}(X) \subseteq X$ is false for every set X. Do not use the Axiom of Regularity.

proof. Assume $\mathscr{P}(X) \subseteq X$ is true, for contradiction. By definition of a powerset, $X \in \mathscr{P}(X)$. By definition of $\subseteq$, since $X \in \mathscr{P}(X)$, $X \in X$. Consider the standard version of Russell's paradox: $R = \{x \mid x \notin x\}$. It informs us that sets that contain themselves are not well defined, as it leads to logical contradictions. Formally, the logical contradiction that follows is: $R \in R \leftrightarrow R \notin R$. Given $X \in X$ and $R \in R \leftrightarrow R \notin R$, we obtain $X \notin X$, a contradiction about X. Therefore, $\mathscr{P}(X) \nsubseteq X$. ☐

Q2. Let R be a binary relation. The domain and range for a binary relation are the same as for a function: $\mathrm{dom}(R) = \{x : \exists y((x, y) \in R)\}$ and $\mathrm{ran}(R) = \{y : \exists x((x, y) \in R)\}$. Prove that the domain and range exist.

proof. Note the definition of a binary relation:

Let A and B be sets. A (binary) relation R from A to B is a subset of $A \times B$.

Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written:

$x \, R \, y$, if and only if, $(x, y) \in R$.

Assume that R is a binary relation from A to B. It immediately follows from the definition of a binary function that the domain and range of R exists. By definition of R, we have (x, y) in $A \times B$. This means that for all x, there will exist a y such that $(x, y) \in R$. Hence, dom(R) is satisfied. Similarly, ran(R) is satisfied as by definition of R, we have (x, y) in $A \times B$, implying that for all y, there exists an x such that $(x, y) \in R$. ☐

Q3. If f is a function and A is a set, must $f \cap A^2$ equal $f \restriction A$? Here $\restriction$ is function restriction.

proof. Consider the definition of a function, $f: X \mapsto Y$, where X and Y are sets, and X is the domain, and Y is the codomain. Note that f can be represented as a set of ordered pairs.

case 1. $A \not\subseteq X$, $A \not\subseteq Y$. By definition of f, $(x, y) \in f \mid x \notin A, y \notin A$. Thus, $f \cap A^2 = \varnothing$. Similarly, $f \upharpoonright A = \varnothing$ as $A \not\subseteq X$ as the domain of f is restricted to $\varnothing$.

case 2. If $A \subseteq X$, $A \not\subseteq Y$, then:

> **case 2.1.** $(x, y) \in f \mid x \in A, y \notin A$. We obtain $f \cap A^2 = \varnothing$ as $y \notin A$.
>
> **case 2.2.** $f \upharpoonright A$ restricts the domain of f, while retaining the codomain of f. Thus, $f \upharpoonright A$ is nonempty, given $A \subseteq X$.

case 2 (rejoinder). Given that $f \cap A^2 = \varnothing$ and $f \upharpoonright A$ is nonempty, $f \cap A^2 \neq f \upharpoonright A$ (as $f \upharpoonright A \subsetneq f \cap A^2$ as there exists an element $z \in f \upharpoonright A$ such that $z \notin f \cap A^2$).

case 3. If $A \not\subseteq X$, $A \subseteq Y$, then $f \cap A^2 = \varnothing$, and $f \upharpoonright A = \varnothing$. The reasoning is similar to case 1. By the definition of f, $(x, y) \in f \mid x \notin A, y \in A$. Therefore, $(x, y) \notin A^2$. Therefore, $f \cap A^2 = \varnothing$. $f \upharpoonright A = \varnothing$ as $A \not\subseteq X$ as the domain of f is restricted to $\varnothing$.

case 4. If $A \subseteq X$, $A \subseteq Y$, then $(x, y) \in f \mid x \in A, y \in A$. Thus, $f \cap A^2 = f \upharpoonright A$ as every element in the LHS and RHS are in A^2.

Rejoinder. Thus, $f \cap A^2$ and $f \upharpoonright A$ need not be equal, as demonstrated in case 2. Restrictions only affect the domain of a function, leaving the codomain unaffected. This is in contrast with the intersection of A^2, which requires elements in the ordered pair to be in A. $\square$

Q4. Prove that if a, b are sets, then (a, b) exists.

proof. Consider the definition of an ordered pair: $(x, y) = \{x, \{x, y\}\}$. Consider the axiom of pairing: $\forall x \forall y \exists z((x \in z) \square (y \in z))$. Assume that a, b are sets. Thus, by the axiom of pairing, we can construct the set $\{a, b\}$. Applying the axiom of pairing again on our newly constructed set, along with a once again, we obtain $\{a, \{a, b\}\}$, which is (a, b). $\square$

Q5. Prove that every vector space has a basis.

definition (vector space). An $\mathbb{R}$-vector space is a set V, with an operation +, and a set of functions $\{f_r: r \in \mathbb{R}\}$ which satisfies some axioms. That is, V contains the 0 vector, is closed under addition, and scalar multiplication.

definition (basis.) A basis of a subspace is a set of linearly independent vectors that spans the subspace.

definition (chain). a sub-ordering $C \subseteq P$ on which $(C, \leq)$ is linear.

definition (maximal). An element p is maximal iff there is no q strictly greater.

Zorn's lemma. Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

proof. $V = \{v_1, v_2, \ldots, v_n\}$ where every v_i is a vector.

case 1. Suppose $V = \{0\}$. It immediately follows that the 0 vector spans V, and that 0 is linearly independent.

case 2. Suppose $V \neq \{0\}$. **w.t.s.** that chains can be constructed, and that every chain has an upper bound. The maximal element is a basis of V.

construction of chains. V is linearly dependent. Therefore, many subsets of V contain vectors which are linearly independent. Let us group all these subsets together. $P = \{W: W \subset V \wedge W \text{ is linearly independent}\}$. We also want P to be partially ordered. How can P be partially ordered? One way is by inclusion, i.e. $W_1 < W_2$ iff $W_1 \subset W_2$. Thus, we order P as such. Given the order of P, there exists chains within P. Thus, $C \subset P$.

upper bound within chains. Given our construction chains, it is not immediately apparent whether there is an upper bound. Suppose C is a particular but arbitrary chain in P. $C = \{c_i\}$ for some indexing i. The upper bound of C can be taken via the axiom of union. $c_{up} = \cup\{c_i\}$ for all i. We could index i with the ordinals. Once we reach an appropriate ordinal, i.e. when all c_i has been included, c_{up} is an upper bound. Note that c_{up} is in C as c_{up} is linearly independent. It also adheres to the partial order as $c_i \subset c_{up}$.

Application of Zorn's lemma. Since P is a partial order, and every chain in P has an upper bound, P contains a maximal element. Take the maximal element to be a basis. ☐