You, me, and mathematical structures

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In this essay, I argue that infinite structures exist, thereby making noneliminative structuralism viable. I argue that since we possess knowledge of theorems, and we are able to discuss mathematics, such structures must exist.

Structuralism argues that mathematics is the study of structures. Shapiro (2000) defines a system as "a collection of objects with certain relations among them" (p. 259). He defines a structure as the "abstract form of a system, highlighting the interrelationships among the objects" (p. 259). In particular, one disregards any non-essential features (i.e. the features that have no influence on relations) about systems when abstracting to a structure. Shapiro defines implicit definitions as the "simultaneous characterization of a number of items in terms of their relations to each other" (p. 283). An example of an implicit definition is the equation $x^2 + y^2 = 1$ where the function itself is implicitly defined by the equation. That is, the equation implicitly defines y as a function of x, without constructively saying what the function is. Another interpretation is that at least two "entities" are being simultaneously characterised, the equation and the function, in terms of their relations to each other. The former is canon, while the latter is more "radical". The latter alludes to the ontological concerns of entities. What exactly is "x" and "y"? What is its mode of being? This is Shapiro's motivation when he makes an ontological distinction about universals: ante rem and in re. He defines ante rem realism as "some universals exist prior to and independent of any items that instantiate them" (p. 262). Contrastingly, in re universals are "universals [that] are ontologically dependent on their instances" (p. 262).

The significance of Shapiro's distinction of ante rem and in re becomes apparent when one considers the ontology of numbers. Suppose we are an in re universalist, and we erased every symbol corresponding to the number 3. We would have successfully destroyed the universal 3. Thus, in re universalists are forced to conclude that 3 no longer exists, and we ought to preserve at least one copy of the symbol in a secure warehouse deep underground. If I were to propose this to the government, I would not only be berated for suggesting the waste of taxpayers' money, but also for suggesting that numbers themselves can be destroyed. Similarly, if I brought this proposal to a mathematics department, they would swiftly escort me out. But what if an in re universalists insists that their conclusion is genuine and true—that numbers can be genuinely destroyed? Consider the closed interval, $[0, 2^{136279841}-1]$. The largest, currently known prime number, which happens to be a Mersenne prime, is so unfathomably large that, reasonably, no one has written it down, prior to the discovery of its primeness. Would it have existed prior to this? An in re universalist would argue no. If that is true, then how could have someone written it down? To acknowledge the social aspect, suppose the first person to state $2^{136279841}-1$ is a bot, or a computer, or someone/something that is not a human. Then the in re universalist would argue we (humans) only learnt of the Mersenne prime because a computer told us. The question remains, how did they, whoever they are, learn or know or write of something that does not exist? The burden is on in re universalists to elaborate. It should be noted that fictionalists have something to say; they argue that mathematical entities should be treated as fictional entities, and that the practice of mathematics itself should be done under pretense, i.e. that we should all pretend that mathematical entities exist. Thus, fictionalists would argue that we either created $2^{136279841}-1$, as we would with fictional entities, e.g. Harry Potter, or that $2^{136279841}-1$ does not really exist. I argue that fictionalists would need to say more, as they have yet to address the necessarily truth aspect of $2^{136279841}-1$ being a Mesenne prime. That is, we know that Harry Potter is a

British alumnus of Hogwarts, for example, but he could have easily been an alumnus of a different school, because we will it so. Since Harry Potter is created by us, would we not have the ability to change things, or make things true or false? Fictionalists could rebut saying that if Harry Potter is no longer an alumnus of Hogwarts, he would not be the Harry Potter, merely a different Harry Potter. While this is fine for fictional characters, is it really the case for mathematical entities, whose properties are necessarily true?

Furthermore, consider Cantor's diagonalisation on the closed interval. Regardless of how many numbers we write down, we will always have a well-defined procedure of generating numbers not already on our list. Thus, in re universalists (and everyone else) are committed to this: one has the ability to generate, and thereby write down, an infinite number of numbers. There is nothing immediately wrong with this ability. Only when coupled with the need for an explanation for how one can write a non-existent object into existence does it become problematic. Similarly, fictionalists would need to say where do the mathematical properties of fictional entities arise from. That is, what makes the n-th term on our list different from our n+1-th term? Thus:

1. We have the ability to discover new theorems.
   1.1. We have the ability to write down an unending number of numbers.
2. We have the ability to discuss mathematical theorems.

In re universalists might counter, arguing that we are not actually writing numbers down, merely symbols which correspond to some number. Thus, when we erase all mathematical symbols, we are not actually destroying anything but the symbols themselves. If in re universalists took this position, they would need to provide an ontology of numbers. Thus far, we have seen the ontological concerns pertaining to mathematical entities. How is structuralism relevant?

In the canon of philosophy of mathematics, some assert that mathematical knowledge is necessarily true, a priori, and abstract (Linnebo, 2017, p. 3). Following the definition of a system and a structure, one must accept one characteristic: abstractness. The status of the remaining two characteristics is not immediately apparent. Structuralism features in the discussion of ontology because of its two perspectives: places-are-offices, and places-are-objects. The places-are-offices perspective is of "the context of one or more systems" (Shapiro, 2000, p. 268). Consequently, "the positions of a structure are more like properties than objects" (p. 268). Contrastingly, the places-are-objects perspective is only in the context "in which the places of a given structure are treated as objects in their own right, at least grammatically" (p. 268). That is, we are sensitive to the particulars of said object, and said structure. To elaborate on the places-are-objects perspective, Shapiro illustrates with a chess analogy (p. 268-9). We distinguish between bishops on black tiles and white tiles, as bishops on black tiles will only ever land on and move onto black tiles, and never on white tiles. Thus, there is a meaningful difference between bishops on black and white tiles, within a specific structure. Whereas, the places-are-offices perspective would be more focused on how bishops on black tiles would behave, across different structures, e.g. across different types of chess boards.

These two perspectives offer us a toolkit for understanding mathematical objects. That said, a problem is made precise from these perspectives; Is the natural number 1 the same as the real number 1? More precisely: are offices of different structures the same? Is the office of 1 in the naturals the same as the office in the reals? Under places-are-objects, one could argue that yes, $1^{\mathbb{N}}$ is the same as $1^{\mathbb{R}}$. I can take a bishop from one chess board to another and nothing about the object itself would have changed. Thus, places-are-objects could argue that some properties are necessarily true, e.g. $1^{\mathbb{N}}$ is the same object as $1^{\mathbb{R}}$. However, this line of reasoning does not immediately say anything about the necessary truth of theorems. For example, "Φ_1: the four colour theorem is true" is necessarily true. This is not to say we could have been mistaken. Rather, the truth value of the four colour theorem is not up to us. My point is: the places-are-objects perspective could explain why certain statements are necessarily true, but whether this is generalisable to all mathematical statements remains to be seen. Likewise for the places-are-offices perspective. There are two immediate objections to this bishop-moving argument. If I moved a bishop from a twelve-by-twelve board to another twelve-by-twelve board, I would not have shifted the bishop from one structure to another. It remains in the same structure, just on a different board. This is easily remedied. I can simply shift the bishop from a twelve-by-twelve to a two-by-two, or a five-dimensional (with time travel) chess board. One can argue that the change in the physical dimensions of a chess board necessitates a change in the properties of a structure. That is, since the three given chess boards are not isomorphic to each other, they are not the same. The second objection is this: how is this analogy relevant to the places-are-offices perspective, given that the mathematical mode of being is not the same as chess pieces? More to the point: what is the mode of being of mathematical objects? Consider this: 1 is the multiplicative identity of any structure. Thus, since only one object has this property, it is the same between structures. A places-are-offices defender would disagree with this justification, arguing that the stated properties (multiplicative identity, uniqueness) are properties of the office itself. The fact that only one object can hold this office is merely a result of the uniqueness property of the office. They could argue, in response to places-are-objects, that the natural numbers have multiple, set-theoretic definitions. Thus, while the underlying constructions could change, the properties of the numbers do not. How might places-are-objects respond? Thus, I argue (trivially) that an important distinction arises. We can have knowledge about properties, without arriving at an ontology. Thus, structuralism is the study of structures, agnostic to an ontology of objects.

Is this agnosticism a genuine part of structuralism, or are structuralists committed to an ontology? Eliminative structuralists argue that they are not committed to the existence of structures, but rather appeal to a background ontology which facilitates the emergence of structures (Shapiro, 2000, p. 271; Linnebo, 2017, p. 164). An example of an appeal to a background ontology is Dedekind's Categoricity Theorem, where he argues that arithmetic statements are generalisations over simply infinite systems (Linnebo, 2017, p. 165). Meanwhile, noneliminative structuralists are committed to the existence of structures, sensitive to the background ontologies it emerges from (p. 161). Dedekind defines "a set S and function s to be a “simply infinite system” if S is one-to-one, there is an element e of S such that e is not in the range of s (thus making S Dedekind-infinite), and the only subset of S that both contains e and is closed under s is S itself. In effect, a simply infinite system is a model of the natural numbers." (Shapiro, 1997, p. 176). I argue that the eliminativist position is problematic due to the non-commitment to the existence of structures. Shapiro (2000) argues that a structuralist needs an account of when a purported implicit definition succeeds (p. 285). He argues that there are two requirements for a successful implicit definition: the existence and uniqueness condition. He argues that at least one structure satisfies the axioms, and that at most one structure is described. If Shapiro is right, how would eliminative structuralists fulfil these two conditions? Noneliminativists could argue that Shapiro's conditions are vacuously met, as structures do not exist. I counter—what are noneliminativists' mathematical statements about, if not mathematical structures? Shapiro (2000) asserts that mathematics presupposes that satisfiability is sufficient for existence (p. 289). He argues that "ante rem and eliminative structuralists accept this presupposition and make use of it like everyone else, and are in no better of a position to justify it" (p. 289). Thus, adhering to mathematical satisfaction, theorems must be satisfied by some mathematical structure, Γ, or Δ. Strictly speaking, any mathematical theorem, φ, need not presuppose the existence of Γ. One could formally sidestep the existence condition by arguing that if Γ exists, then Γ semantically and syntactically entails φ. However, my question remains: how can we have knowledge of φ, if φ does not actually exist?

3. Let Γ be some mathematical structure. Let Δ be the ZFC axioms. Let $\mathscr{L}$ be an appropriate language of Γ. Let φ be some well-formed formula of $\mathscr{L}$.
4. If Γ exists, then φ is syntactically and semantically entailed by Γ for some valuation, a.
5. We have knowledge of φ.
6. If φ exists, then Γ exists.
7. If I have knowledge of φ, φ exists.
8. Therefore, Γ exists.

Thus, I argue that mathematical knowledge is knowledge of mathematical structures, i.e. knowledge of the existence and nature of structures, its objects, the theorems and proofs that follow, that some conjectures are true, or false, or undecidable, &c. Thus, the contention is about where our ability to generate knowledge arises from.

9. We generate knowledge from Γ.
10. Δ, e.g. the ZFC axioms, is forced on us to be true.
    - 10.1. That is, Δ is necessarily true, and abstract.
    - 10.2. Δ need not be the ZFC axioms. It could be any suitable set of axioms.
11. First-order logic is necessarily true, and abstract.
12. We have the ability to create mathematical definitions.
13. Set theory need not be the foundation of mathematics. It could suitably be category theory.

Thus, I argue that the suitable background ontology really is mathematical structures. Thus, we have reason to be noneliminative structuralists. Eliminativists structuralists would disagree, as they are only committed to the patterns which arise from the natural world. But what are these patterns, if not instances of mathematical structures? Thus, the necessarily true aspect arises from (9.)-(11.). Linnebo (2017) characterises this commitment as the ontological dependence claim (p. 163). I see nothing wrong with being ontologically dependent on Δ, or $\mathscr{L}$, or first-order logic. In contrast, Resnik (1981) argues that mathematical objects are "structureless points" without "internal compositions" which are somehow "arranged in structures", or are just "positions in structures" (Risnek, 1981, p. 530; Linnebo, 2017, p. 162; Shapiro, 2000, p. 259). As Linnebo and Shapiro notes, Resnik's position is problematic. What does it mean for relations to exist independent of structures? How can something be arranged in a non-existent structure? Noneliminativists need not be strictly committed to the premises (7.) to (13.). For example, Shapiro (2000) takes issue with (9.), and builds his structuralist position on higher-order logic (p. 267). Thus, reconstructions of this nature are possible. Similarly, we need not be committed to simply infinite structures, as we can replace Dedekind's simply infinite structures with mathematical structures in set-theory, e.g. $\mathscr{N} = (\mathbb{N}, (f_i)_{i \leq n}, (R_j)_{i \leq n}, (c_k)_{i \leq n})$, which is assigned the smallest cardinal, ${}_0$□—i.e. $\mathscr{N}$ is an infinite structure. Minimally, noneliminativists are committed to some mathematical structure being true, be it set theory, or category theory, or something else. I argue that this commitment is not problematic, as we have the ability to do mathematics. There is some degree of freedom in our practice and philosophy of mathematics as we have the ability to "create" definitions, e.g. $\mathscr{L} = ((f_i)_{i \leq n}, (R_j)_{i \leq n}, (c_k)_{i \leq n}, =, <)$. However, nothing is false about the definitions that we use. Thus, this degree of freedom is not problematic. Furthermore, we understand our theorems. We can prove other theorems with it, and we can apply it outright. Mathematical meaning exists because there is an underlying truth about the relations we describe. To take a leaf out of Maudlin (2019)'s book, why can't mathematical objects have its own, unique ontology, beyond Aristotle's and our current classifications of being, thereby providing an answer to the ontology of mathematics (p. 89)? More could be said about the commitment to the existence of structures. For example, could we not also generate knowledge from the possible existence of structures? I am skeptical of this, as then we would be committed to our theorems being possibly true, not necessarily true. Thus, I have shown that mathematical knowledge is generated from mathematical structures. Since we have knowledge-of, such structures genuinely exist, which makes our commitments non-problematic.

References

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